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GEOMETRY

FOR

GRAMMAR SCHOOLS

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OF NEW YORK

JANUARY 25, 1924

GEOMETRY

FOR GRAMMAR SCHOOLS.

BY

E. HUNT, LL.D.,

FORMERLY HEAD-MASTER OF THE GIRLS' HIGH AND NORMAL SCHOOL,
BOSTON; NOW SUPERINTENDENT OF SCHOOLS FOR
WINCHESTER AND MEDFORD.

* * *

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PREFACE.

A GREAT lover of geometry used to remark, "without the geometrical faculty we should be idiots"; that is, unable to differentiate the forms of objects, we could not tell one from another.

In the kindergarten and the lowest primary grade the child is introduced to geometry in art, and there seems to be no good reason why in the sixth, seventh, and eighth grades he should not be introduced to the science of geometry, in its elementary concepts, definitions, and methods.

Traditional methods of studying geometry, and the difficulty of selection from the ordinary text-book, stand in the way of its introduction into the grammar school.

Teaching by dictation has been carried to excess in all branches, and the Socratic method, originally used to refute erroneous opinions, may easily degenerate with pupils who have no opinions of the subject, correct or erroneous, into —

> "Letting down buckets into empty wells
> And growing old in drawing nothing up."

To learn from the printed page is a most valuable part of the pupil's education. For this reason the pupil is expected to make the diagrams for the text, unaided by the teacher.

In geometry, as in other branches, the pupil will learn only what he wants to know. To implant this want in the pupil's mind skilfully, and at the same time not to tell him what with proper effort he can find out for himself, distinguishes superior teaching.

I wish to thank the publishers for the great pains they have taken to have errors in the copy and proof corrected, and also all who have kindly read the proof, and made valuable suggestions and corrections.

E. H.

PART I.

LESSON I.

To go from one place or point, *A*, to another place or point, *B*, we start from *A*, and move all the time towards *B*, without changing our direction, if possible. All animals, including insects, will do the same thing. They seem to know as well as we do, that a line that does not change its direction — that is, a straight line — is the shortest distance between two points.

What is a straight line?

A **straight line** is a line that never changes its direction, or, It is the shortest distance between two points.

Only one straight line can be drawn between two points.

The ends of a line are called **points**, and the length of a line is the distance between these two points.

3 dimension
length.
breadth & thick
meas. ille.
1 dim.

A **line** has only length; no breadth, no thickness.

A **point** has no length, no breadth, no thickness. – no "

Points are only places where lines begin or end, or cross each other. Emphasize. Points everywhere. Space may be filled with points.

The points and lines that we make with the pencil and crayon are not the real points and lines we talk about in geometry: they only show us where the real points and lines are; that is, right in the middle or the center of the dots and lines that we can see.

Indicate that we may [illegible] a point line or [illegible] [illegible] where no [illegible] visible.

LESSON II.

2 dimensions) A **surface** has length and breadth, but no thickness. It is simply the outside of bodies.

When the surface is flat or level, like the blackboard, top of the table, or the side of a slate, it is called a **plane surface.**

In geometry, we mean by a **plane,** a surface so even that a straight line drawn from any point in it to any other point in it will be wholly in the plane; that is, will touch it at every point.

The word *line*, used alone, means a straight line.

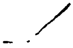
We can imagine a line made by a point moving in a plane. All the lines we shall talk about in the first part of this geometry are supposed to be drawn in the same plane.

only 2 dimensions in plane geometry.

Two or more lines having the same direction are called **parallel lines.**

Two lines drawn so as to make a square corner, or angle, are **perpendicular** to each other.

Two lines, having different directions not perpendicular, are **oblique** to each other.



If a line be drawn in a plane, any other line, drawn in the same plane, must be oblique, perpendicular, or parallel to the line first drawn.

A line parallel with the horizon or bottom of the page we call a **horizontal line.**

A vertical line is the direction of a plumb line.

Draw several kinds of lines. Test straightness with ruler.

Place two points; measure the distance between them. Draw a short line; another; one equal to both.

LESSON III.

Draw the different kinds of lines, 2 inches long; 3 inches; 4 inches. (Freehand.) Name them.

Draw three short lines; one equal to the three.

Draw four lines, 2 feet long, $\frac{1}{2}$ inch apart; $\frac{1}{4}$ inch; $\frac{1}{8}$ inch.

Draw four parallels on the blackboard, 3 feet long, 2 inches apart; 3, 4, 5, 6 inches apart.

To draw parallel lines on slate and paper, use the ruler and the graduated right angle or square, 3 inches by 4 inches. At the same time the pupil learns that two points determine the direction of a line.

To draw parallels on the blackboard, use a square, 2 feet by 3 feet.

The right angle and ruler should be graduated to sixteenths of an inch on one side, and to millimeters on the other.

[Draw lines as above, using decimeters, centimeters, millimeters, until the pupil is familiar with the names and the distances they represent.]

Give a card, having lines of different lengths, to each pupil. Let him measure them by each scale. Compare results.

LESSON IV.

To read a line, we use two letters.

Name the starting-point or origin first, as *AB* or *BA*.

In comparing lines as to length, say one is so many times the other, or that they are to each other, as 2 to 3, 5 to 7, as the case may be.

We do not write an inch as the fraction $\frac{1}{12}$ of a foot, but as a whole number, the unit of measure. So, in measuring short lines, we can use the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ inch as the unit of measure.

Draw a line $\frac{7}{8}$ of an inch long. What distance would the line represent, if $\frac{1}{8}$ represent 10 feet? 20 rods? 5 miles?

Draw a line $1\frac{1}{2}$ inches long. What distance will it represent on a scale of 20 feet to an inch? 10 rods to $\frac{1}{2}$ inch?

Draw a line to represent 60 rods on a scale of 10 rods to $\frac{1}{2}$ inch.

Draw a line 100 feet long, scale 10 feet to the $\frac{1}{4}$ inch.

Draw a line 100 meters long, scale 10 meters to a centimeter.

Draw two lines; measure and compare them.

LESSON V.

Estimate, and then measure: length of lines drawn on the blackboard, horizontal, vertical, and oblique.

(1) Short lines in inches, centimeters, etc.

(2) Longer lines in feet and decimeters.

Each pupil makes careful record of estimate, or guess, before testing.

Estimate and test: length and width of slate, desk, pane of glass; width of aisle, door, schoolroom; height of, and other objects. Average estimates of class.

Also, estimate distances in yards, meters, rods, and miles.

Let each pupil find his own height; distance he can span with arms outstretched; span of thumb and forefinger; width of hand and length of forefinger; especially, length of step in natural gait, taking the average of several steps.

A line composed of two or more straight lines is called a **broken line.** Draw one from *A* to *B*; also a straight one.

Measure them; see which is the shorter.

What is a straight line?

Make and name the different kinds of lines.

No time will be lost, if two or three weeks are put upon this important lesson.

LESSON VI.

When two lines, *AB*, *AC*, are drawn from the same point in different directions, they make an **angle.**

The point of meeting is **the vertex of the angle.**

The difference in direction of the two lines is **the measure of the angle.**

If there is but one angle at the point, we can read it with one letter; as, $\angle A$. ($\angle$ is read 'angle.')

If there are two or more angles at the same point, we must use three letters to read one of them; thus, $\angle BAC$, $\angle CAD$, etc., being careful to place as the middle letter the one at the vertex of the angle.

When two lines, *AB*, *CD*, do not meet, we can prolong them until they do meet, if possible; if not, draw a line from *A*, parallel to *CD*, and the $\angle A$ is their difference of direction.

When a vertical line meets a horizontal line, what kind of angles are made?

When they cross each other, how many right angles are made at the point of intersection?

When a line makes right angles with another, one is said to be perpendicular to the other.

All lines, not parallel, are oblique or perpendicular.

Draw and name the different kinds of lines.

LESSON VII.

When a point moving in a plane constantly changes its direction, it traces a curved line, or **curve.**

A straight line can meet or cut another straight line in one point only.

It can cut a curve in at least two points. Try it.

What is a curve?

It is a line that constantly changes its direction, or it is a line that can be cut by a straight line in at least two points.

The most useful and common curve is the **circle.**

Every point in it is equally distant from a point within it, called the **center.**

It is very easy to describe a circle with the compasses, or with a string and crayon or pencil.

We have only to make a point move around another point which is fixed, keeping always at the same distance from the fixed point; the moving point describes a circle.

The space enclosed by the circle is also called a circle.

A line from the center to any point in the circle is called a **radius**; as, *Ca*, *Cb*, etc.

Name objects of a circular form. All the lines you know.

LESSON VIII.

The plural of *radius* is *radii.*

A part of the circumference, as *ab*, is called an **arc.** It is the measure of the angle made at the center of the circle by the radii *Ca*, *Cb*.

Let *BAC* be any angle; *A* is the vertex, *AB* and *AC* are the sides.

The length of the sides makes no difference in the size of the angle.

The sides do not change their direction.

An angle is measured by the arc of a circle intercepted by its sides.

The circle may have any radius whatever, but its center must be at the vertex of the angle.

For convenience in measuring angles, every circle is divided into 360 equal parts, called **degrees**, written thus, 360°.

A degree, then, is simply the 360th part of a circumference.

Draw a circle and a horizontal line *PCA* through the center, *C*. Draw the vertical line, *CQ*.

What part of the circle is intercepted by the radii, *CA* and *CQ?*

How many degrees measure the $\angle ACQ$?

What kind of an angle is an angle of 90°?

How many degrees in the $\angle ACD$? 45°. $\angle ACB$? $22\frac{1}{2}°$. Draw them.

LESSON IX.

Angles less than 90° are **acute angles.** *Acute* means *sharp.*

Make several acute angles; estimate the degrees they will measure, then test them with the protractor. (Pupil must be told how to make protractor and how to use it.)

An angle greater than a right angle is an **obtuse angle.**

$\angle ACR$ is 135°. Make and estimate several obtuse angles.

To estimate an obtuse angle, imagine or mark off the right angle; then estimate the acute angle left, and add it to 90°.

CP makes an angle with *CA* of 180°, or two right angles. That is, the opposite of any direction takes half of a circle, or 180°, sometimes called **a straight angle.**

What is a right angle? An acute? An obtuse?

For greater accuracy of measurement, the degree is divided into 60 equal parts, called **minutes,** written $60'$. The minute is divided into 60 equal parts, called **seconds,** written $60''$.

Draw several radii of a circle.

Name two angles that together make a right angle. Three.

Name the angles that together make two right angles.

When a straight line meets another straight line so as to make two angles, the angles are called **adjacent angles.**

LESSON X.

A vertical line meets a horizontal line, making two adjacent angles; what is each angle called? What is their sum?

When an oblique line meets a straight line, what is the sum of their adjacent angles? Prove it.

A vertical line crosses a horizontal line; how many angles are made; and what are they?

When two lines intersect obliquely, on each side of each line are **supplementary angles.** Name them all.

Two angles are supplementary when their sum is 180°.

What must the sum of all the angles made about a point be?

What must be the sum of all the angles on one side of a line, the angles having their vertices at the same point?

When two lines intersect, the opposite angles are called **vertical angles.**

Vertical angles are equal angles. See if you can prove it. Number the angles, 1, 2, 3, 4.

Angle 2 and angle 1 make 180°. Angle 2 and angle 3 make 180°. What can you say, then, of angles 3 and 1?

LESSON XI.

Draw a right angle. Divide it by a line from the vertex.

Each of the angles, *BAC*, *CAD*, is the **complement** of the other.

To find the complement of an angle, subtract it from 90°.

To find the supplement of an angle, subtract it from 180°.

Draw an acute angle. Make its supplement. (Continue either side beyond the vertex.)

Draw an acute angle. Make its complement. Draw a line through the vertex ⊥ to either side. (⊥ is read 'perpendicular.')

Make an angle. Make another equal to it. (Vertical angles.)

What is the supplement of an angle of 25°? 75°? 100°?

What is the complement of an angle of 20°? 45°? 60°? 22½°?

Define and illustrate:

A right angle.	Vertical angles.
An acute angle.	Adjacent angles.
An obtuse angle.	Supplement of an angle.

Complement of an angle.

Two angles at the same vertex, having one side common, are adjacent angles, but not necessarily supplementary or complementary adjacent angles; to be such, their sum must be 180° or 90°.

LESSON XII.

Draw two intersecting lines; mark the equal angles, 1, 1, 2, 2, or with colored crayons.

Draw two other intersecting lines; mark the angles, *a*, *b*, *c*, *d*.

Write equal angles thus:

$$\angle a=\angle\quad;\ \angle b=\angle\quad;\ \angle c=\angle\quad;\ \angle d=\angle.$$

Draw three oblique lines through a point; mark equal angles, 1, 1, 2, 2, 3, 3.

Draw the same and mark the angles, *a*, *a*, *b*, *b*, *c*, *c*.

Again draw the same; mark the angles in order, *a*, *b*, *c*, *d*, *e*, *f*.

Express the equality of the vertical angles, thus:

$$\angle a=\angle\,;\ \angle b=\angle\,;\ \text{etc.}$$

Write the sum of two angles equal to the sum of two other angles, all possible cases, thus:

$$\angle\text{s}\, a+b=\angle\text{s}\quad+\quad;\ \angle\text{s}\, b+c=\angle\text{s}\quad+\quad;\ \text{etc.}$$

Write the sum of three angles equal to the sum of three other angles, all possible cases:

$$\angle\text{s}\, a+b+c=\angle\text{s}\quad+\quad+\quad.$$

Express that the sum of three of the angles is equal to 180°, all possible cases.

If the sum of any two of the angles having a common side is 100°, what is the measure of the angle on either side of them?

If any angle is 45°, what is the sum of the two angles next to it?

Change the position of the lines, letters, and numbers, in several examples, until the equal angles are easily recognized.

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LESSON XIII.

Draw a line intersecting obliquely two parallel lines (called a **secant line**, or **transversal**).

How many angles are formed?

Since the parallels have the same direction, how many different angles are formed?

Letter the angles, *a*, *b*, *c*, *d*, and *e*, *f*, *g*, *h*.

Name four angles equal to each other. Four others.

For convenience of reference, these angles are called:

If within the parallels, **interior angles.**

If without the parallels, **exterior angles.**

Read the exterior angles; the interior angles.

If on the same side of the secant line, not adjacent, any two angles are called **opposite angles.**

On each side of the secant line, we have, then:

Opposite interior angles. Name them.

Opposite exterior angles. Name them.

Opposite exterior and interior angles. Name them.

Two angles, not vertical, not adjacent, and not on the same side of the secant line, are called **alternate angles.** Name them.

Alternate angles are either exterior, or interior, or both.

Any two of the angles in this lesson are equal, or else their sum is 180°.

Tell what is true of all the different kinds of angles, and why.

LESSON XIV.

Draw two parallel lines.

Cross them with two oblique parallels. How many different angles?

Number them. Letter them. Compare them as before.

Draw parallels as before, cross them with two parallels, and a fifth line through two points of intersection.

How many angles? Number equal angles as before.

Sum of the angles, 1, 2, 3?

Letter them all with different letters.

Begin with *a*, and write all the angles equal to *a*.

All equal to *b*, *c*, etc.

Sum of angles, *a*, *b*, *c*?

Sum of the angles in each triangle?

Write the angles equal to two others taken together.

LESSON XV.

Make an obtuse angle, sides 1 inch long. A right angle, sides 2 inches long.

Which is the larger angle?

Make a small acute angle. A large one.

Draw line *AB* 2 inches long. At point *A* make a large acute angle, that is, one more than 45°, with line $2\frac{1}{2}$ inches long, *AC*.

Join *CB*. What do you call the figure *ABC*?

How many sides has a triangle?

How many angles? What kind of angles are *A*, *B*, and *C*?

The triangle *ABC* is, therefore, called an **acute triangle.**

What is an acute triangle?

An acute triangle has all its angles acute.

The line *AB*, on which the triangle rests, is its **base.**

We can imagine it to rest on either side, then, called the base.

Draw a line *AB* 2 inches long.

At *A* make an angle a little larger than a right angle, $AC=1$ inch.

Join *BC*. What kind of an angle is *A*?

Then the triangle is called an **obtuse triangle.**

What is an obtuse triangle?

An obtuse triangle has one obtuse angle.

LESSON XVI.

Make an acute triangle. (Freehand.)

Make an obtuse triangle. (Freehand.)

Draw AB 2 inches. BC vertical, shorter than AB. Join AC. What kind of an angle is B?

Then the $\triangle ABC$ is called a **right triangle.**

$\triangle$ is read *triangle.*

What is a right triangle?

A right triangle has one right angle.

The line AB is the base. What is the base of a triangle?

The line BC is the perpendicular. C is the vertex of the triangle.

The base of a triangle is the side opposite the vertex.

The line AC is the hypotenuse. It is opposite what angle?

The side opposite the right angle is always the hypotenuse.

If we call BC the base, what is the perpendicular? Hypotenuse?

Try to make a triangle with two obtuse angles.

A triangle with one obtuse angle and one right angle.

One with two right angles.

Can there be more than one right angle in a triangle?

LESSON XVII.

Make an acute angle with 2-inch lines AB and AC.

Join BC for the base.

Because AB and AC are equal, $\triangle ABC$ is called an **isosceles triangle.**

What is an isosceles triangle?

An isosceles triangle has two equal sides.

If the isosceles triangle is an acute triangle, it may be called an **acute isosceles triangle.**

Make an obtuse angle with two equal lines AB, AC.

Join BC. What kind of an isosceles triangle is it? An **obtuse isosceles triangle.**

Make a right triangle, base and perpendicular equal.

An **isosceles right triangle.**

Cut an isosceles triangle out of paper; fold it on a line through the vertex and the middle point of the base.

Compare the angles at the base.

In a triangle, the angles opposite the equal sides are equal.

The equal angles of an isosceles triangle are always acute. Why?

Make (freehand) rapidly and name one of each kind of triangle you can remember.

LESSON XVIII.

Draw a circle with radius of 1 inch.

Through *C* draw *ECA*. *EA* is a **diameter** of the circle.

With compasses open to radius, begin at *A* and walk them around the circle, finding the points *B*, *D*, *E*, *F*, *G*. Join *C* to each point.

Into how many equal parts, or angles, is the circumference divided?

What part of 360° is $\angle BCA$?

How many degrees? Draw line from *A* to *B*.

A line joining two points of the circumference is called a **chord.**

Then *AB* is the chord of 60°.

Compare *AB* with *CA*. Are they equal? Therefore, the chord of 60° is equal to radius. Of any circle? Try it.

Draw 2-inch line *AB*. With *A* and *B* as centers, and radius *AB*, describe arcs, intersecting in *C*. Join *AC* and *BC*.

What can you say of the sides of the $\triangle ABC$?

Then it is an **equilateral triangle.**

What is an equilateral triangle?

An equilateral triangle has its sides equal.

What can you say about its angles?

An equilateral triangle is equiangular.

Draw line *AB* with *A* as center, and different radii; describe several arcs. Lay off angles of 60° on each arc. Does the length of the radius make any difference in the size of the angles?

LESSON XIX.

Make two unequal acute angles, *A* and *B*. Which is the greater?

With *A* and *B* as centers, and radius less than the shortest side, cut the sides of the angles with the arc of a circle.

Which angle stands on the larger arc?

Make a triangle, sides 2 inches, 2½ inches, and 3 inches.

See if you can make another different triangle, with the same lines.

Only one triangle can be made with the same three lines.

The triangle has no two sides equal, and therefore it is called a **scalene triangle.**

Define a scalene triangle.

A scalene triangle has no two sides equal.

Observe: The greater side is opposite the greater angles.

Prove it. Measure the arcs of a circle with same radius, from each vertex.

Make a paper triangle of the same size; compare the angles by folding.

The longest side is shorter than the sum of the other two. Why? A straight line is the shortest distance between two points.

A scalene triangle may be an acute triangle.

Can it be a right triangle? An isosceles? Why not?

Can it be equiangular? Why not?

Draw, freehand, and name all the different kinds of triangles.

LESSON XX.

Make two equal angles about 50° each; side *AB* of one, and *DE* of the other, $2\frac{1}{2}$ inches; sides *AC* and *DF* 2 inches each. Draw *BC* and *EF*.

The ⊿s *ABC* and *DEF* are equal in all respects. Because,

(1) We have made two sides and the included angle of one equal to two sides and the included angle of the other, each to each.

(2) Put the vertex *D* upon vertex *A*; side *DE* upon side *AB*, and as they are equal, the point *E* will fall upon *B*.

As the angles *A* and *D* are equal, the side *DF* will take the direction *AC*, and *F* will fall upon *C*, because *AC* and *DF* are equal.

EF and *BC* must then coincide; for only one straight line can be drawn between two points.

The angles opposite the equal sides are equal. Therefore,

CASE I. When two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the remaining parts are equal, each to each, and the triangles are equal.

Make any △ *ABC*. Extend the sides *AB* and *CB*, making *BE* equal to *BC*, *BD* equal to *BA*. Join *DE*.

Prove *DE* is equal to *AC*. (Case I.)

LESSON XXI.

Make a $\triangle ABC$. Draw DE equal to AB. At D make an angle equal to $\angle A$. At E make an angle equal to $\angle B$. Sides intersect at F.

The $\triangle DEF$ will be equal to the $\triangle ABC$. For,

(1) We have made two angles and the included side of one equal to two angles and the included side of the other, each to each.

Place DE upon AB. As $\angle D$ equals $\angle A$, DF takes the direction AC, and F must fall in the line AC.

As $\angle E$ equals $\angle B$, EF takes the direction BC, and F must fall in the line BC.

As F can be in both lines, AC and BC, only at their intersection, C, the angles are equal.

$DF=AC$, $EF=BC$. Why? Therefore,

CASE II. When two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the remaining parts will be equal, and the triangles are equal.

Make any $\triangle ABC$. Extend sides AB, CB. On extension of AB, take $BD=AB$.

(1) From D draw a parallel to CA, intersecting CB, produced at E.

Are the triangles equal? Why? (Case II.) Name equal parts.

(2) Make any $\triangle ABC$. At A and B make $\angle BAD = \angle BAC$, $\angle ABD = \angle ABC$.

AB separates the triangles. Are the triangles equal? Why? Name equal parts.

(3) Make any $\triangle ABC$. At *B* make $\angle ABD = \angle ABC$. Take $BD = BC$. Join *AD*.

Are the triangles equal? Why? Name the equal parts.

With base, *AB*, 3 inches, make any kind of a $\triangle ABC$. If *AB* represents 120 feet, yards, or rods, how many feet, yards, rods, will one inch represent? 40, which is called the **reducing factor.**

How many to ½ inch? ¼ inch? ⅛ inch?

And 20, 10, and 5 are reducing factors for their scales.

Distance of *C* from *A* and *B*? Make a $\triangle ABC$. $AB = 60$ meters. Find *AC*, *BC*.

◆

LESSON XXII.

Surveyor's compass, signal staffs, chain, tape, marking rods, and how to use them, can be explained best with the objects in hand. Pupils can furnish most of them, good enough for practical purposes.

To find a perpendicular to a line in the field, nail to top of a staff a block 4 inches square, 2 inches thick, with fine saw cuts through its diagonals.

For angles of elevation, on side at top of staff, fix a graduated 4-inch circle.

A strip 18 inches by 1 inch by ½ inch, bearing an index needle, all centered with circle.

For horizontal angles, the same nailed to top of staff, arm bearing sights.

Teach pupil to set staff vertical with plumb-line.

In solving the following problems, the pupil should be given ample time before receiving assistance.

PROBLEM 1.

To find the distance between two objects, one being inaccessible, *B*.

This is to be solved by the pupil in the field, if possible, or in the schoolroom.

Suppose the two objects are *A* and *B*, a pond intervening.

Take some point *C* visible to both objects.

(1) Find the angle *CB* makes with *CA*.

(2) Place a staff at *C*, and at a point *E*, fixing the direction of a line *CX*, making the same angle with *CA* that *CB* does.

(3) Find angle *AB* makes with *AC*.

(4) Find direction of line *AY*, making same angle with *AC* that *AB* does. In line *AY* place a staff at *D*, also in line with *EC*.

(5) Measure distance from *A* to *D*, and the problem is solved. Why?

PROBLEM 2.

In the same field find the distance of *G* from *C*, visible from *A* and *C*, and on the same side of *AC* as *B*.

LESSON XXIII.

PROBLEM 3.

Find the distance between two inaccessible objects, *C* and *D*, both visible from two points, *A* and *B*.

(1) Begin with *A*, and with *A*, *B*, *C*, find a point *B*, on side of *AB* opposite *CD*, as in Problem 1.

(2) At *B*, with *B*, *C*, *D*, find *F* in the same way.

$EF = CD$. *EF* is the only line to measure.

Let pupil write out the steps taken.

PROBLEM 4.

An inaccessible object *B* is visible from *A* and *X*, and the distance from *A* to *X* is 40 meters; find *AB* and *XB*.

We can measure *AX*, but on account of hill and woods we cannot do the field work as before. We must plot it on paper.

Draw *AX*, 40 meters to any scale.

At *A* we found the $\angle BAC$, 75° 30′. On our plan we draw an indefinite line *AY*, making with *AX* an angle of 75° 30′.

At *X* we find $\angle AXB$ is 40°; draw the indefinite line *XZ*, making $\angle AXZ = \angle AXB$. *XZ* and *AY* will intersect at *D*. $AD = AB$. $XD = XB$. Both can be measured to same scale as *AX*.

PROBLEM 5.

Find distance between two inaccessible objects, both visible from *A* and *B*, 60 rods apart.

No field work is possible, except taking all the angles as in Problem 3. We must plot it. Draw *AB* to any scale, find *EF*, measure it to the same scale.

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LESSON XXIV.

Draw a right △*ABC*. Extend base to *E*.

The ∠*EBC*, formed outside of the triangle by producing a side, is called the exterior angle of the triangle.

With same radius at each angle, describe arcs at *A* and *C*, measuring them, and a semi-circumference at *B*, including angles at *B*.

With the compasses, see if the sum of the arcs at *A* and *C* is equal to arc of exterior angle.

Cut a paper right triangle. Fold the vertices on the right angle.

Do they both together make a right angle?

Then, what is the sum of the three angles of a right triangle equal to? 180°, or two right angles.

Draw an equilateral △ *ABC*. Produce the base to *E*.

Strike arcs at the vertices in the triangle as before; on arc of exterior angle take the chords of the non-adjacent angles.

In this case we know the exterior angle is equal to the sum of the two non-adjacent angles, for the angles of the triangle are each 60°, and the exterior angle is 120°.

Draw a scalene triangle. Produce one side as before.

Angle at vertex of isosceles triangle is 47° 50′, angle at the base is .....?

Angle at base of isosceles triangle is 72° 40′, angle at vertex is .....?

An angle of a scalene triangle is 37° 50′, sum of the other two is .....?

Sum of two angles of a triangle is 124° 57′, the third is .....?

Pupils should make many similar questions for class to solve.

LESSON XXVII.

The relation of 2 to 4 is the same as the relation of 1 to 2. That is, 2 : 4 :: 1 : 2, which is read, *two is to four as one is to two.*

Again, 3 : 6 :: 5 : 10; this is evidently true, for $\frac{6}{3}=2$, and $\frac{10}{5}=2$, and we say 3 and 6, 5 and 10, have a common ratio.

That two numbers have a common ratio with two other numbers, is generally expressed as above:—3 : 6 :: 5 : 10, and the expression is called a **proportion**; for convenience of reference the numbers are named in order, the first, second, third, and fourth terms of the proportion.

A proportion, then, simply expresses the fact that two numbers have the same ratio as two other numbers, or it is the equality of two ratios.

In the proportion, 3 : 6 :: 5 : 10, observe that $6 \times 5 = 3 \times 10$.

That is, the product of the second and third terms is equal to the product of the first and fourth.

That is true of every proportion, and is the test of the correctness of the proportion.

The product of two numbers is 30; one is 3, what is the other?

If we have 3 : 6 : : 5 :   , how shall we find the fourth term?

Make the rule to find the fourth term when you have the first, second, and third.

Try several examples with small numbers.

LESSON XXVIII.

Write the numbers, 4, 6, 8, 12, in the form of a proportion, in every possible way, testing the correctness of each form. How many forms?

The first and fourth terms of a proportion are called the **extremes**; the second and third, the **means.**

If we have 5 : 10 : : 6 :   , how do you find the fourth term?

What is the rule?

Divide the product of the means by the first term.

If 3 : 6 : : 15 : 30, is 3 : 15 : : 6 : 30 correct?

Can we write it 15 : 30 : : 3 : 6? Then the means and extremes can change places.

$$\tfrac{15}{3}=5, \quad \tfrac{30}{6}=5;$$

the ratios are equal; 30 and 6 are called the **terms of the ratio.**

A ratio is the quotient of one number divided by another of the same kind.

It is evident we cannot compare dollars with miles, lines with surfaces, etc. The terms of a ratio must represent quantities of the same kind.

If we multiply both terms 30 and 6 by the same number, whole, mixed or fractional, will the ratio be changed? Try it.

If we divide them by the same number, will the ratio be changed? Try it with several examples.

Therefore, we conclude, we can multiply or divide both terms of a ratio and not change its value.

If we have the proportion $\frac{1}{2} : \frac{1}{3} = 6 :$.

Multiplying terms of first ratio by 6, $3 : 2 = 6 : 4$.

LESSON XXIX.

In arithmetic, proportion is made a rule for solving problems. The truth is, it is the only method by which all practical problems are solved.

All persons, educated or uneducated, use proportion in solving questions, can use no other method.

The solution of the simplest problems, involving multiplication or division, is always by proportion.

Thus, if 2 apples cost 3 cents, what will 4 cost? Apples and cost must have a common ratio, and we say, 6 cents.

We do not write it out in set form, but use the method, $2 : 4 :: 3 : 6$.

Again, if 1 apple costs 2 cents, what will 3 cost?

1 : 3 :: 2 : 6.

Or, if 2 apples cost 6 cents, what will 1 apple cost?

2 : 1 :: 6 : 3.

Though such questions are said to be done by analysis, it is really by proportion, the only rule by which we do practical questions in arithmetic after we have learned notation, numeration, addition, and subtraction.

It is of the utmost consequence, then, that we become familiar with proportion, since it holds the same importance in arithmetic that similar triangles do in geometry.

In fact, the problem of similar triangles is only the application of proportion to geometrical questions.

LESSON XXX.

Draw two parallel lines intersecting obliquely two other parallels. Draw secant through two intersections.

Prove that the two triangles, made by the secant within the parallels, are equal.

And therefore, that two parallels intercept equal portions of two other parallels.

There are six parts of a triangle: three sides and three angles.

Triangles are *equal*, if their sides are equal, each to each.

Triangles are *similar*, if their angles are equal, each to each.

We will now talk about the greatest, most useful, if not the simplest, of all geometrical problems,—The comparison of similar triangles.

Surveyors, engineers, navigators, and astronomers have books containing the solutions of all possible triangles set down in tables especially made for them, and when they have a triangle to solve, that is, to find some part of it, they first find three parts of their triangle, one of which must be a side, and then compare it with the similar triangle solved in the tables.

Cut three small equal right triangles. Place them so as to make a large hollow right triangle of exactly the same shape as the small triangle.

The angles of the large triangle are exactly the same as the angles of the small triangle.

The base of the large triangle is exactly twice the base of the small triangle.

Perpendicular of large triangle is twice the perpendicular of the small triangle.

Hypotenuse of large triangle is twice the hypotenuse of the small triangle.

LESSON XXXI.

Place three more triangles of same size on the perpendicular side of the large triangle. The form or shape of the large triangle is exactly the same as before. The angles are all equal to the angles of the small triangle, each to each. But,

Base of large triangle is three times base of small triangle.

Perpendicular of large triangle is three times perpendicular of small triangle.

Hypotenuse of large triangle is three times hypotenuse of small triangle.

Continuing to add successively four, five, six, etc., small triangles to the same side, we shall always have large triangles whose sides are four, five, six, etc., times the corresponding sides of the small triangle.

The small triangle and the large triangles have the same angles, and their corresponding sides are to each other as 2, 3, 4, 5, 6, etc., and they are called **similar triangles.**

If we take small equilateral, isosceles, or scalene triangles, we can build up the large triangles in the same way.

What are similar triangles?

Similar triangles have all their angles equal, each to each, and their corresponding sides have *a common ratio.* Or,

Similar triangles have all their angles equal, each to each, and the sides about the equal angles proportional.

LESSON XXXII.

Draw a right triangle. Base 4 inches, perpendicular 3 inches. Measure hypotenuse; scale, 6 feet to an inch; sides equal .....?

At $\frac{1}{2}$ inch from angle at the base draw a perpendicular to the base.

Is the small triangle similar to the whole triangle?

Base of small triangle represents how many feet?

Base of large triangle is how many times base of small triangle?

Perpendicular and hypotenuse of the large triangle will be how many times perpendicular and hypotenuse of small triangle? Why?

At every half-inch in base draw a parallel to perpendicular.

We have four triangles with common vertex. Letter or number them, and compare sides of any two of them, if they are similar, to different scales.

PROBLEM.

To find height of flag-staff.

First method. Place a stake vertically at suitable distance from the staff. Find some point on the ground, *C*, in line with top of staff and stake. Divide distance from *C* to foot of staff by distance from *C* to foot of stake. The quotient is the common ratio. Multiply the height of stake by the ratio, and you have height of staff. Why?

LESSON XXXIII.

The first method of finding the height of an object can easily be applied in the schoolroom.

Second method. When an object casts a shadow, it is easy to make a good approximate calculation of its height, near midday.

Pace the shadow of the object, and compare it with your own shadow, knowing your own height. Or,

Stick a knife into the object at a height you can measure or estimate, and compare the distance of the knife's shadow from the foot of the object with the shadow of the object.

The first and second methods can be used when the object stands on a level or an inclined plane.

Third method. If the object stands on a level plain, at some point *C*, take the angle of elevation of top of the object, make an equal horizontal angle, the side intersecting a line from the object perpendicular to a line from *C* to the object. Perpendicular distance from object to intersection is the height required.

Let pupils estimate heights of neighboring objects, trees, schoolhouse, spires, etc., and afterwards measure them. Compare results.

Much time can be profitably given to this problem.

LESSON XXXIV.

The corresponding sides of similar triangles are called **homologous sides.**

The pupil cannot become too familiar with the proposition, that the homologous sides of similar triangles have a common ratio, or are proportional.

Equal triangles are only particular cases of similar triangles, where the common ratio of the homologous sides is unity, or 1.

PROBLEM.

The distance between two accessible objects, *A* and *B*, cannot be measured directly on account of intervening objects. Find it when *A* and *B* are visible to each other.

With two straight edges two feet long, nailed together at an angle of 60° standing at *A*, fix the direction of a line from *A* making an angle of 60° with *AB*.

At *B* fix the direction of another line making an angle of 60° with *BA*.

The distance of the intersection *C*, of these two lines from *A* or *B*, is the distance sought.

Standing at *C*, the straight edges should point to *A* and *B*. Why?

It must not be forgotten, that one example made and performed by the pupil will be worth more to him than any number simply explained by the teacher, however ably done.

In addition to accurate field work, and work with compasses and ruler, there should be much freehand work, thereby cultivating the imagination of the pupil.

LESSON XXXV.

Draw an equilateral triangle.

Draw a line from vertex to middle point of base.

Are the two triangles formed equal?

Are the two angles at the middle point of the base equal?

What kind of angles are they?

What kind of a line, then, is the line from the vertex?

As the greater side is opposite the greater angle, which is greater, the side of the triangle, or the line to middle point of base?

Then, a perpendicular is the shortest distance from a point to a line.

To divide a line into two equal parts.

With radius AB, strike intersecting arcs on each side of AB. Join the points of intersection.

To divide a line AB into two or more equal parts.

Draw AC making any angle with AB. On AC take equal distances, AE, EF, FG, etc.

Join last point with B. Draw parallels through E, F, G, etc. And AB is divided into equal parts.

Draw a perpendicular from a point to a line AB.

With the point as center, describe an arc cutting the line in two points. With these points as centers, describe arcs intersecting on side opposite the given point.

Draw line from point to intersection.

Draw a perpendicular from a point in a line.

Bisect a line.

If two parallels meet or cross two other parallels at right angles, the space included by the four parallels is called a **rectangle.**

LESSON XXXVI.

The opposite sides of a rectangle are equal. Why?

Draw a square. How many right angles has it?

How does a diagonal divide it?

What, then, is the sum of angles of either triangle?

Draw a rectangle. How many right angles has it?

How does a diagonal divide it?

What is the sum of the angles of either right triangle?

Sum of the acute angles of a right triangle equals .....?

Any right triangle may be considered as half the rectangle of the same base and altitude.

Draw a perpendicular from vertex of an equilateral triangle.

Degrees in each angle? Sum?

Draw a perpendicular from vertex of an isosceles triangle.

What can you say of the triangles? Sum of angles?

Draw any scalene triangle with perpendicular from vertex on the base.

Prove that the sum of the angles of the whole triangle is equal to two right angles.

Therefore the sum of the angles of any triangle is equal to two right angles.

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LESSON XXXVII.

QUADRILATERALS.

A four-sided figure is called a **quadrilateral.**

Draw a figure with four unequal sides. (Fig. 1.)

How many angles has it?

Are any of its sides parallel?

It is a **trapezium.**

What is a trapezium?

A trapezium is a quadrilateral with no two of its sides parallel.

Draw a quadrilateral with only two of its sides parallel. (Fig. 2.)

How does it differ from Fig. 1?

It is a **trapezoid.** Define a trapezoid.

A trapezoid is a quadrilateral with only two of its sides parallel.

Draw a quadrilateral with opposite sides parallel. How does it differ from Figs. 1 and 2?

Figure 3 is a parallelogram.

Define a parallelogram.

A parallelogram has its opposite sides parallel.

A quadrilateral is read with four letters at the vertices, *a*, *b*, *c*, *d*, or two opposite letters, *ac* or *bd*.

A line joining *ac* or *bd* is called a **diagonal.** How does it divide the parallelogram?

The two diagonals mutually bisect each other.

As the opposite sides of a parallelogram are parallels intercepted by parallels, how do they compare?

Draw and name the quadrilaterals you know.

LESSON XXXVIII.

How many angles in a parallelogram?

If each angle is a right angle, it is called a **rectangle.** (Fig. 4.)

Define a rectangle.

A rectangle is a right parallelogram.

Name some rectangular objects you can see.

If the sides of the rectangle are all equal (Fig. 5), what is it?

Define a square.

A square is a rectangle with all its sides equal.

If a parallelogram has an acute angle (Fig. 6), it is an oblique parallelogram. Compare Figs. 1, 2, 3, 4, 5.

It is called a **rhomboid.**

Define a rhomboid.

A rhomboid is a parallelogram having no right angle.

If the rhomboid has all its sides equal (Fig. 7), it is called a **rhombus.**

Define a rhombus.

A rhombus is a rhomboid with all its sides equal.

Draw, name, and define all the quadrilaterals you have learned.

Encourage pupils to vary their definitions of the different geometrical figures and propositions, criticising each other's, many of which will be mathematically correct, and help them to decide upon the best as to accuracy and brevity.

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LESSON XXXIX.

How many kinds of quadrilaterals can you make?

If the opposite sides of a quadrilateral are equal, what do you call the figure?

How many kinds of parallelograms are there?

Draw a quadrilateral having three other names.

What are the four names?

Draw one with two names. With three names.

Draw any parallelogram.

What of its opposite sides?

Opposite angles?

What is the sum of any two consecutive angles?

How does a diagonal divide it?

How do both diagonals divide each other?

The perimeter of a figure is the distance around it, or the sum of its sides.

What must be known to find the perimeter of a square? Of rhombus?

One side of a square or rhombus is 5 feet, perimeter is .....?

What must be given to find perimeter of rectangle or rhomboid?

One side is 6 feet, adjacent side 4 feet, perimeter is .....?

What must be known to find perimeter of trapezium or trapezoid?

Give examples.

Quadrilaterals having all their angles equal, each to each, taken in the same order are similar, if their homologous sides have a common ratio. Pupil can prove this, drawing diagonals and comparing triangles.

LESSON XL.

In Algebra, numbers are represented by letters, and operations by signs as in arithmetic.

Thus, $2a$ means 2 times a, a representing any number whatever.

If the value of a is 5, the value of $2a$, $3a$, etc., will be

$$2a = 2 \times 5 = 10;$$

$$3a = 3 \times 5 = 15, \text{ etc.}$$

$$a + 5a = 6a. \qquad 8a - 3a = 5a.$$

$$3a + 4a = 7a. \qquad 7a - a = 6a.$$

An expression of the equality of two quantities is an **equation.**

The quantity on one side of the sign = is a **member.**

Quantities connected by the sign plus (+), or minus (−), are **terms** of the equation.

The number before a letter is its multiplier or coefficient.

When a is 6, and b is 8,

$$4a + 2b = \ ? \qquad 3a - b = \ ? \qquad 3b - 4a = \ ?$$

$$a \times a = a^2, \qquad a \times a \times a = a^3.$$

The figure at the right and above a letter is its **exponent,** and shows how many times the letter is to be used as a factor.

If a is 5, $\quad a^2 = 5 \times 5 = 25.$

$$a^3 = 5 \times 5 \times 5 = 125.$$

a^2 is read, *a second power, or a square.* a^3 is read, *a third, or a cube.*

If a is 4, and m is 6,

$$2a^2 = \quad ? \qquad 4m^2 = \quad ? \qquad 2m^2 - a^3 = \quad ?$$

Let pupils make and solve many other examples.

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LESSON XLI.

The lengths of lines may be represented by letters, a, b, etc.

The product of two lines by two letters, thus, ab, ac, bc, etc.

That is, a may represent any number of units of length in a line, inches, feet, yards, rods, miles, meters, etc.; b another, and c another number.

ab means a times b. If a is 6 and b 5, $ab = \quad ?$

Let a, b, and c represent the sides of a triangle; then, $m \cdot a$, $m \cdot b$, and $m \cdot c$ will represent the sides of all possible similar triangles, if we regard m as a multiplier of a, b, and c, and not as a line.

$$\frac{m \cdot a}{a} = m, \qquad \frac{m \cdot b}{b} = m, \qquad \frac{m \cdot c}{c} = m.$$

The sides are homologous because they have a common ratio.

Therefore, if a, b, and c represent the sides of any plane triangle that can be imagined, then $m \cdot a$, $m \cdot b$, $m \cdot c$ will represent all possible similar triangles.

If a, b, and c are 4, 5, and 6, and m is 3, what will be the sides of the similar triangle?

Give m any value, whole, mixed, or fractional, and find other similar triangles.

If $a=b=c$, what kind of a triangle would they represent?

LESSON XLII.

$$\frac{4\times5}{4}=5, \qquad \frac{ab}{a}=b.$$

$$\frac{4\times5}{5}=4, \qquad \frac{ab}{b}=a.$$

That is, dividing the product of two factors by one of them gives the other.

If $ab=p$, then $a=\frac{p}{b}$, and $b=\frac{p}{a}$.

$$\frac{4\times5\times6}{4\times5}=6, \qquad \frac{abc}{ab}=c, \qquad \frac{abc}{bc}=a, \qquad \frac{abc}{ac}=b,$$

which tells us, dividing the product of three factors by the product of two of them gives the third factors

If $abc=p$, then $a=\frac{p}{bc}$, $b=\frac{p}{ac}$, $c=\frac{p}{ab}$.

Let $2pr=c$, then $r=\frac{c}{2p}$.

If $Dr=C$, then $r=\frac{C}{D}$.

The square root of a^2 is a, written thus:

$$\sqrt{a^2}=a.$$

If $a^2=9$, then $\sqrt{a^2}=a=\sqrt{9}=3$;

or, $$a^2=9.$$

Then $$a=3.$$

Again, $$pr^2=a,\ r^2=\frac{a}{p},$$

and $$\sqrt{r^2}=\sqrt{\frac{a}{p}};$$

or, $$r=\sqrt{\frac{a}{p}}.$$

$a+b$ means the sum of any two quantities.

$2a+2b=2(a+b)$, that is, the sum of $2a$ and $2b$ equals twice the sum of a and b.

See if it is true when $a=4$ and $b=5$.

$ac+bc=c(a+b)$. Verify it when $a=4$, $b=5$, and $c=6$.

LESSON XLIII.

A point moving in one direction describes a straight line.

Now, imagine the whole line to move in a direction perpendicular to its own direction.

What kind of a figure would it describe?

The space passed over by the line is called the **area** of the rectangle.

How must the line move to describe a rhomboid?

Make a square, side 1 inch. It is a square inch.

When we ask how many square inches in any surface, desk, slate, pane of glass, etc., we mean how many of these little squares would it take to cover it.

Draw a rectangle 2 inches by 1 inch.

How many square inches in its surface?

Make a square, side 2 inches.

How many square inches in its surface?

What is the difference between 2 square inches and 2 inches square?

Difference between 3 square inches and 3 inches square? Prove by figure.

How many square inches can you place in a rectangle 6 inches by 4 inches?

How many rows? How many in a row? Square inches in its area?

How do you find the area of a rectangle?

Multiply the length by the width.

Square inches in a rectangle 2 feet long and 6 inches wide?

Let pupils make many examples.

LESSON XLIV.

Side of a square is 6 inches, area is .....?

Multiplying a number by itself is called **squaring it.**

If the sides were given in feet, yards, meters, rods, etc., then our unit of measure would be a square foot, square yard, square meter, etc.

Find the number of square inches in a square foot.

Square feet in a square yard. Square inches in a square yard.

Make the table of square measure.

How many square inches in your slate? Desk? Pane of glass?

Find square feet in door, floor, ceiling, sides, and ends of the room.

Find area of school lot, common, or field, in square yards, rods, meters.

If the sides of a rectangle are a and b, its area equals ab.

Having the area and one side given, how is the other side found?

The perimeter equals $2(a+b)$, or $2a+2b$.

A has a lot 20 rods square. B has one 25 rods by 16 rods.

Which has the most to pay to fence his lot at $1 a rod? How much?

Area of square field is 625 square rods, one side equals .....?

Pupils should learn to extract the square root to two or three places.

How does a diagonal divide a rectangle?

The altitude of a triangle is the perpendicular distance from its vertex to the base.

The altitude of a rectangle is its width.

LESSON XLV.

From the vertex of an obtuse triangle draw a perpendicular to the base.

Construct a rectangle on the same base, including the triangle, and having the same altitude.

The triangle is what part of the rectangle?

If we call the base b, the altitude a, and the area p, we have for the rectangle,

No. 1. $p=ab$. No. 2. $a=\frac{p}{b}$. No. 3. $b=\frac{p}{a}$.

Nos. 1, 2, and 3 are called **formulæ**: that is, they are the algebraic expressions for finding the parts of a rectangle. Translated or interpreted,

No. 1 means, Area is equal to product of base and altitude.

No. 2 means, Altitude equals area divided by base.

No. 3 means, Base equals area divided by altitude.

For the triangle we have,

No. 1. $p=\frac{1}{2}ab$. No. 2. $a=\frac{p}{\frac{1}{2}b}$. No. 3. $b=\frac{p}{\frac{1}{2}a}$.

Or, $a=\frac{2p}{b}$, and $b=\frac{2p}{a}$.

See if you can translate the formulas into English.

Draw a rhombus with both diagonals.

Prove that the diagonals divide the rhombus into equal parts, and that the area of the rhombus is one-half of the product of the diagonals.

Diagonals of rhombus are 4 and 6; what is the area?

LESSON XLVI.

Figures having all their sides and angles equal, each to each, are **equal figures.**

Figures whose areas only are equal are **equivalent figures.**

What are similar figures?

Name some of each kind.

Draw a parallelogram with oblique angles. Its name?

Cut a paper rhomboid. Draw a perpendicular from the obtuse angle at the upper base to the lower base. Cut off the right triangle and add it to the other end of the rhomboid, so as to make a rectangle having the same base and altitude.

It is evident the area of the rhomboid is the same as that of a rectangle with same base and altitude.

The altitude of a parallelogram is the perpendicular distance between its bases.

The area of any parallelogram is the product of its base and altitude.

PROBLEM.

If two parallels cross two other parallels at any angle whatever, the four parallels will always include the same area. Prove it.

Which will have the least perimeter?

Parallelograms having equal bases and altitudes are equivalent.

LESSON XLVII.

Draw a trapezoid. The nonparallel sides are called its **legs.** Draw a perpendicular to its parallel bases. Bisect the legs by a line drawn parallel to the bases. It is called the **median line** of the trapezoid.

Prove that it is half the sum of the bases.

Area of the trapezoid is the product of the median and altitude.

Ex. Parallel sides are 13 and 17, altitude 6; area of trapezoid is .....?

Draw a trapezium standing on the longest side.

Make a trapezoid and a triangle of it by a line drawn parallel to the base. Let fall a perpendicular from vertex of triangle to base of trapezium. What lines must be known to find the area?

Or, divide trapezium by a diagonal into two triangles. Draw their perpendiculars.

Pupils make examples for solution.

Plane figures with three or more sides are called **polygons.**

If the sides are unequal, they are irregular polygons; and to find their area, divide them into trapezoids and triangles.

If the sides and angles of a polygon are equal, it is a regular polygon.

They are named:

with 3 sides, a trigon;
4 sides, a tetragon;
5 sides, a pentagon;
6 sides, a hexagon;
7 sides, a heptagon;
8 sides, an octagon;
9 sides, a nonagon;
10 sides, a decagon;
11 sides, an undecagon;
12 sides, a dodecagon.

LESSON XLVIII.

How many trigons, or equilateral triangles, can be placed about a single point?

How many tetragons or squares? Pentagons? Hexagons? Octagons? Dodecagons?

If a floor is to be laid with tiles of a single form, what regular polygons may be used? Why?

What combinations of other geometrical figures may be used?

How many circles can be drawn about a single point?

Distance apart of the centers of two circles touching each other externally?

Draw three equal circles touching each other externally.

Join their centers; what kind of a triangle is formed?

Draw three circles with different radii, but having the same center.

They are concentric circles.

Draw two circles cutting each other in two points.

The line joining the points of intersection is a common chord perpendicular to their radii.

LESSON XLIX.

A lot of land is 1 rod square. What is the distance around it, that is, its perimeter?

Make the square.

Make another, 2 rods square.

What is its perimeter?

What is the perimeter of a square, 3 rods on a side? 4 rods?

Make all these squares on a small scale in a row, and write their perimeters under them, thus:

4 8 12 16.

Dividing by 4, the perimeters are as

1 2 3 4,

that is, the perimeters are as the sides.

A few examples with similar triangles, rectangles, and other similar figures, will enable the pupil to discover that the perimeters and the homologous sides of similar figures have a common ratio.

Or, the perimeters of similar figures are to each other as any two of their homologous sides.

Perimeters of regular polygons are as their apothems.

Perimeters of circles are as their radii.

Perimeters of similar figures are to each other as any lines similarly drawn in each.

Pupils must see clearly that perimeters are only distances, and so are properly compared with lines.

LESSON L.

Draw a scalene triangle resting on its longest side.

Let fall a perpendicular on the base. What kind of triangles are formed?

Draw another with homologous sides twice as long as the sides of the first.

A perpendicular from the vertex to the base divides this triangle into two right triangles.

Are the right triangles of the second similar to the right triangles of the first? Why?

Draw a square, side 2 inches.

Another, side 3 inches.

Their areas are as $2^2 : 3^2$, or $4 : 9$.

Divide each square into two equal rectangles.

Are they similar?

Since each rectangle is half of its square, the areas of the similar rectangles are to each other as the squares of their homologous sides. Or,

The squares and the similar rectangles must have a common ratio.

Draw a diagonal in one of the rectangles of each square. Are the triangles similar?

Since the triangles are halves of similar rectangles, and all triangles can be divided into right triangles, their areas must be to each other as the squares of their homologous sides.

Since similar surfaces can be divided into the same number of similar triangles, we conclude that,

Areas of similar surfaces are to each other as the squares of any two homologous sides: or, the squares of any two lines similarly drawn in them.

LESSON LI.

Areas of regular polygons are to each other as the squares of their apothems.

Areas of circles are to each other as the squares of their radii.

Area of a rectangular lot, 10 rods long and 6 rods wide, equals .....?

Area of a rectangular lot, 30 rods long and 18 rods wide, equals .....?

Are the above rectangles similar?

What is the common ratio of their homologous sides? Ratio of their areas?

The ratio of the areas of similar surfaces is the square of the common ratio of their homologous sides or corresponding lines.

Draw a circle, 2-inch radius. Another, 3-inch radius.

Ratio of their circumference is .....? 2:3; which means that the first is two-thirds of the second, or the second is three-halves of the first.

Ratio of their areas is .....? $2^2:3^2$, or 4:9; which means,

The first is four-ninths of the second, or the second is nine-fourths of the first.

Take the number 10.

Divide it into equal parts,

$5 \times 5 = 25$, a square.

Divide it into unequal parts,

$4 \times 6 = 24$, a rectangle;

or $3 \times 7 = 21$, a rectangle;

or $2 \times 8 = 16$, a rectangle.

How do the perimeters of the above square and rectangles compare? Their areas? Which shows that of rectangles with equal perimeters, the square has the maximum area?

LESSON LII.

To divide a triangle into 2, 3, 4, or more equal parts, by lines from the vertex,

Divide the base into 2, 3, 4, or more equal parts, and draw lines to the points of division. Why?

The areas of triangles having equal bases are to each other as their altitudes.

The areas of triangles having equal altitudes are to each other as their bases.

The areas of triangles having equal bases and equal altitudes are equal.

Triangles having their base in the same line and their vertex in a line parallel to the base, must have the same altitude. Why?

Draw an obtuse triangle ABC, obtuse angle at B; let fall a perpendicular from C on the base produced. Through C draw XY parallel to base.

At middle point *D* of base, draw a perpendicular *DE*.

Suppose the vertex *C* moves along *XY* to *E*, what kind of a triangle have we then?

As it moves from *E* towards *X* or *Y*, do the sides increase or decrease?

With the same base *AB*, two lines drawn from *A* and *B* intersecting anywhere in *XY*, always inclose the same area. Why?

Which of the triangles has the least perimeter?

The two lines might be drawn to intersect at the same distance on the other side of *AB*.

We have two lines, then, parallel to *AB*, at any point in which two lines may be drawn to *A* and *B*, and the triangles so formed will all have the same area. Why?

LESSON LIII.

Where can you place a point in a plane and always have it at the same distance from a fixed point? Therefore, we say,

The circumference of a circle is the locus of a point that is always at the same distance from another fixed point.

In like manner we can say,

Two parallel lines are the locus of the intersection of two lines drawn from the extremities of a given line so as always to include the same area. Why?

Ex. 1. I wish to inclose 40 square rods in form of a triangle, the base to be 10 rods long. Where must the vertex be to take the least fence?

Ex. 2. A triangular lot contains 1000 square feet; one side is 50 feet long. What is the length of the homologous side of a similar triangle containing 4000 square feet?

Ex. 3. A rectangular field contains 100 acres; one side is 80 rods long. How many acres in a similar field, with homologous side twice as long?

The surfaces of spheres are similar, and they are to each other as the squares of their circumferences, or as the squares of their diameters.

Ex. 1. A ball is 3 inches in diameter; another is 6 inches. How do their surfaces compare?

Ex. 2. Call the earth 8000 miles in diameter, and the moon's diameter 2000 miles. How do their surfaces compare?

LESSON LIV.

Draw a right triangle, base 4 inches, perpendicular 3 inches. Measure the hypotenuse. It is 5 inches.

Make squares on the legs and hypotenuse of the triangle.

Divide each of the squares into square inches.

See if the sum of the square inches on the legs is equal to the square inches on the hypotenuse.

It ought to be so, because

$$3^2+4^2=5^2, \quad \text{or} \quad 9+16=25.$$

Therefore, to make a right angle, such as the carpenter uses to make a square corner, we can nail together two straight edges so that a point 3 feet on one edge, and another 4 feet on the other edge, from the angle, shall be just 5 feet apart. Or 6, 8, and 10 feet, such as the builder uses.

Draw a square.

Draw another on its diagonal.

Show that the square on the diagonal is twice the original square.

If we have two equal ratios, say, $\frac{6}{4}=\frac{9}{6}$, then $6^2=4\times 9$.

Again, if we have

$$\begin{array}{ll} & 48=12\times 4\\ \text{and} & \underline{36=12\times 3}\\ \text{then will} & 84=12\times 7\\ \text{or} & 84=12\times(4+3).\end{array}$$

Let the pupil see clearly how 12×4 and 12×3 are added.

Make other examples.

LESSON LV.

If two right triangles have an acute angle in each, equal to each other, all the other angles are equal, each to each, and the triangles are similar.

Draw a right triangle ABC, standing on the hypotenuse AB.

We will call the sides opposite the angles A, B, and C, a, b, and c, respectively.

From the right angle C draw a perpendicular CD to the base AB. It divides the base c into two parts, called **segments.**

The segment about $\angle A$, we will call x, the other y.

Then $x+y=c$.

We now have three right triangles to compare: the large triangle and the two small ones.

Are they similar? Why?

We will consider only the homologous sides about the angles A and B.

Write them in pairs.

The homologous sides of ADC and ABC are b, x, and c, b.

The homologous sides of BDC and ABC are a, y, and c, a.

The homologous sides have a common ratio; then

$$\frac{b}{c}=\frac{x}{b}, \quad \text{or} \quad b^2=c\times x,$$

and

$$\frac{a}{c}=\frac{y}{a}, \quad \text{or} \quad a^2=c\times y.$$

Adding, we have $$a^2+b^2=c(x+y),$$

or $$a^2+b^2=c^2, \quad \text{as} \quad x+y=c.$$

Hence, the square on the hypotenuse is equal to the sum of the squares on the base and perpendicular.

This is called the **Pythagorean proposition.**

LESSON LVI.

If a, b, are the sides, and c the hypotenuse, then

$$c^2=a^2+b^2, \quad \text{or} \quad c=\sqrt{a^2+b^2};$$

that is, the hypotenuse is equal to the square root of the sum of the squares of the sides.

Ex. 1. Sides are 18 and 24; what is the hypotenuse?

Ex. 2. A ladder placed 12 feet from a house reaches a window 16 feet from the ground; length of ladder is .....?

Find diagonal of the room; of the side; of the end. Show that the diagonal of the room is the square root of the sum of the squares of its three dimensions.

Ex. 3. Area of a square field is 400 square rods; length of diagonal .....?

$$a^2+b^2=c^2,$$

then $\quad a^2=c^2-b^2,$ and $b^2=c^2-a^2;$

and to find a and b, we have

$$a=\sqrt{c^2-b^2} \text{ and } b=\sqrt{c^2-a^2}.$$

That is, to find either side,

Extract the square root of the difference between the square of the hypotenuse and the square of the other side.

Ex. 4. Hypotenuse is 100, base is 80; other side is .....?

Ex. 5. How far from a house may a ladder 30 feet long be placed to reach a window 24 feet from the ground?

Pupils can easily make other examples for the class.

Since the areas of similar surfaces are as the squares of their homologous sides or lines, the Pythagorean proposition is true for any similar figures drawn on the sides of a right triangle: similar triangles, rectangles, regular polygons, circles, and semicircles.

Try equilateral triangles and compute their areas.

Ex. 6. The area of a right triangle is 324 square yards, and half of the base is 16 yards; hypotenuse and perpendicular are .....?

LESSON LVII.

Take any three points *A*, *B*, *C*, not in a straight line; draw *AB* and *BC*.

At the middle point of each line draw a perpendicular. They will intersect at *E*; with *E* as a center, and radius *EA*, describe a circle. Will it pass through points *B* and *C*? Prove it.

A circle can always be made to pass through three points not in a straight line.

Therefore, to find the center of a circle when the circumference of an arc is given, Take three points in the arc or circumference; draw chords *AB* and *BC*: the intersection of the perpendiculars from their middle points will be the center.

A perpendicular from the middle point of a chord of an arc passes through the center.

In the same circle equal chords are equally distant from the center.

An angle at the center of a circle is measured by the whole of the arc on which it stands.

An angle with vertex in the circumference is called an **inscribed angle.**

It is measured by half of the arc on which it stands.

Show that is true by inscribing a hexagon, an equilateral triangle, and a square in a circle.

Also give the general proof thus:

From a point in the circumference draw a diameter and a chord; draw a radius to end of chord.

The inscribed angle is half the angle at the center; hence it is measured by half its arc.

If the inscribed angle includes the center, draw a diameter from the vertex, and the proof is obvious.

If the center is outside of the angle, draw the diameter as before, and see if you can prove it.

LESSON LVIII.

Connect the extremities of a diameter with any point in the circumference of a circle.

What kind of an angle is formed? Why?

Two lines drawn from the extremities of a given line on either side of it intersect at right angles. What is the locus of their point of intersection?

Ex. 1. How does a line from the vertex of a right angle to the middle point of the hypotenuse compare with the hypotenuse?

Ex. 2. How does the radius of its circumscribing circle compare with the hypotenuse of a right triangle?

A **secant** is a line cutting the circumference in two points.

A **chord** divides a circle into two segments, a less segment and a greater segment; unless it becomes a diameter dividing the circle into semicircles.

A chord subtends the arc of its segment.

The angle inscribed in a segment is measured by half the arc of the other segment.

All angles inscribed in the same segment are equal. Why?

Then an arc of a circle is the locus of the intersection of two lines drawn from the extremities of a given line, always making the same angle with each other.

Since the lines may intersect on either side of the given line, the locus is a double segment, and the given line is a common chord of two equal intersecting circles.

LESSON LIX.

To draw a perpendicular at B, in line AB.

At C, a little distance from AB, nearer B than A, with radius CB, describe a circle cutting AB in D. Draw DC, cutting the circle in E. BE is the direction of the perpendicular. Why?

Draw any quadrilateral in a circle. Is the sum of its opposite angles equal to two right angles? Why?

When the sum of two of the opposite angles of a quadrilateral is equal to two right angles, it can be inscribed in a circle.

How would you find the center of the circle?

What is the length of the shortest chord that can be drawn through a point 4 rods from the center of a circle, diameter 10 rods?

TANGENTS.

Draw a perpendicular to a radius at any point in the circumference of a circle.

It is called a **tangent**, because it just touches the circle without cutting it. Rather, it is drawn through two consecutive points of the curve, and shows the direction of the curve, which is always at right angles to the radius.

To draw a tangent to a circle from a point outside the circle.

From the center *C*, draw *CA* to a given point *A*. At *B*, middle point of *AC*, describe a circle, with radius *BC*, cutting the circle in *E* and *F*. *AE* and *AF* are the tangents. Why?

Two tangents can always be drawn to a circle from a point without.

LESSON LX.

Draw a circle with any radius.

Draw the chords of 60°.

Name the figure. How many sides has it? Draw radii to one side.

Side subtends an angle at the center of how many degrees?

All regular polygons can be inscribed in a circle.

The center of the circle is the center of the polygon.

To draw a regular polygon, then, step round the circle with the chord of the angle at the center, which is 360° divided by the number of sides of the polygon to be drawn.

The angle made by two adjacent sides of the polygon is the supplement of the angle at the center subtended by a side. Prove it.

Join alternate angles of the hexagon. Side is chord of .....? Angle of polygon is .....? What is the figure?

Draw a polygon with chord of 30°. Name the figure. Angles equal .....?

Show that a polygon can be divided into as many triangles as it has sides, having their vertices at the center.

What is the sum of all the angles at the center?

What is the sum of all the angles of the polygon?

Produce the sides and show that it is as many times two right angles as the polygon has sides less two.

A perpendicular from the center on a side is the **apothem.**

The area of one of the triangles is the side multiplied by one-half of the apothem.

The area of the polygon is the perimeter multiplied by half of the apothem.

◆

LESSON LXI.

AREA OF THE CIRCLE.

Draw a circle. Inscribe in it a hexagon.

Draw two radii to one side. Draw apothem of polygon to the same side.

Call the radius 10. Find length of apothem.

Find area of the triangle. Area = side × ½ apothem.

How many such triangles in the hexagon?

Then, the whole area (A) of the polygon is

$$6 \times \text{side} \times \tfrac{1}{2} \text{ apothem.}$$

But 6 times a side is the perimeter (P) of the polygon; therefore, we have for the area of hexagon,

$$A = P \times \tfrac{1}{2} \text{ apothem.}$$

Produce the apothem to the circumference.

Draw sides and apothem of a dodecagon. Find length of apothem and area of dodecagon. The area of dodecagon is greater than the area of the hexagon.

In like manner, if we increased the number of the sides of the inscribed polygon to 24, 48, 96, and so on, the area of the polygons would be

$$A = P \times \tfrac{1}{2} \text{ apothem.}$$

If we should continue doubling the sides of the polygon, at last the *perimeter* of the polygon would very nearly coincide with the *circumference* of the circle, and the *apothem* would become the *radius* of the circle, and like the polygons, the area of the circle would be

$$A = \text{circumference} \times \tfrac{1}{2} \text{ radius.}$$

Circumference is 3.1416, diameter 1. What is the area?

◆

LESSON LXII.

Circles are similar surfaces, and their circumferences are to each other as their diameters; or the circumferences and diameters have a common ratio.

To find this ratio is a very important problem.

It is the celebrated problem of squaring the circle, which no sane person now attempts to solve. And yet it is no more difficult than finding the square root of 2, which we can imagine perfectly as the diagonal of a square, calling the side 1, but cannot express exactly in figures.

With a slip of writing paper, which is better than a string, measure very carefully the ends of several spools or cylinders of different sizes, and compare their circumferences with their diameters.

You should find the circumference nearly $3\frac{1}{7}$ times the diameter.

More exactly and near enough for all practical purposes, the ratio is found to be 3.1416.

It is called π (Pē), the name of the Greek letter P.

So π always stands for the number 3.1416. $\pi = 3.1416$.

Then for the circle we have, $\pi \times D = \text{circumference}$, and $D = \dfrac{\text{circumference}}{\pi}$.

As $D = 2r$, or diameter equals twice the radius, $2\pi r = \text{circumference}$.

$\text{Area} = \text{circumference} \times \frac{1}{2}r$, or $2\pi r \times \frac{1}{2}r = \pi r^2 = \text{area}$.

$\pi r^2 = \text{area}$, then $r = \sqrt{\dfrac{\text{area}}{\pi}}$. Read these expressions in words.

Ex. 1. Area of circle, radius 6 feet, is what?

Ex. 2. Area, radius 12 feet, is what? Compare with No. 1.

LESSON LXIII.

How many degrees in a circumference?

If the circumference is given, how is the length of one degree found?

Call the circumference of the earth 25,000 miles; what is the length of a degree of latitude? 10°? Width of the temperate zone in miles?

A sector of a circle is a part of its surface bounded by two radii and the arc intercepted by them.

To find the area of a sector, we must know the radius and the angle at the center, or length of the arc on which it stands. Area $=$ arc $\times \frac{1}{2} r$.

Find area of sector, radius 200 feet, angle or arc 30°.

A part of the circle bounded by an arc and its chord is a **segment.**

If we know the length of the chord, radius, and angle at the center, we can find area of the segment.

First find area of the sector of which the segment is a part, from that subtract the area of the triangle formed by radii to the chord.

Draw two concentric circles.

$$\begin{array}{lll} & \text{area of larger} & = \pi R^2, \\ & \text{area of smaller} & = \pi r^2; \\ \text{then} & \text{area of ring} & = \pi R^2 - \pi r^2, \\ \text{or} & & = \pi (R^2 - r^2). \end{array}$$

What is the area of a ring, radius of outer circle 12 inches, of the inner 10?

LESSON LXIV.

What is a tangent?

Are the two tangents to a circle from the same point equal?

Draw any quadrilateral about a circle. Prove that the sum of two opposite sides is equal to the sum of the other two sides.

Ex. 1. Radius of a circle is 6 feet. Find length of a tangent from a point 10 feet distant from the center.

Ex. 2. A boy fastens one end of a rope 20 feet long to a stake; how large a lot can he walk around with the other end?

Of rectangles of the same area, which has the least perimeter?

Area of a circle is 400 square feet, what is the radius? Diameter? Circumference?

What is the perimeter of a square containing 400 square feet? Which has the least perimeter for the same area, the square or the circle?

Of all figures then, we say,

The circle has the maximum area with a minimum perimeter.

Ex. 3. A man having a circular farm of a thousand acres, with his house at the center, agreed to give to each of his three sons a farm if they would divide it as he directed. They must reserve for him the largest triangle that could be made in the circle; their farms must be of the same size and shape, and two touch each other and his at each corner; they must build a house on their farm at the nearest point to his.

How many acres did he reserve? How many did a son receive? Distance of their houses from his and from each other? Shortest distance from their house across their farm? Longest distance on each farm from the house?

LESSON LXV.

If you cast the shadow of a circular hoop or ring on the wall in one position, it will be a circle; revolve the ring slowly on a diameter, and the shadow will be a succession of beautiful curves, growing gradually narrower, until it becomes a straight line.

This flattened curve is an **ellipse.** It is a curve of very great importance in geometry, in art, in engineering, and especially in astronomy; for the earth and other planets, in their revolution around the sun, describe ellipses.

Fasten with pins the ends of a string, a little longer than the distance of the pins apart; with the point of a pencil, keeping the string constantly stretched, you can describe an ellipse.

The points made by the pins are the **foci.**

Midway between the foci is the **center.**

The longest diameter through the center and foci is the **major axis.**

The shortest diameter is the **minor axis.**

All chords through the center are **diameters.**

The distance of a point in the curve from a focus is a **radius vector.**

From the method of drawing an ellipse, it is evident it is a curve, the sum of the distances of any point in which to the foci is always the same.

The sum always equals the major axis.

The distance from the center to a focus is the **linear eccentricity.** When it is zero the ellipse becomes a **circle.**

Draw several ellipses with different eccentricities.

Let semi-major axis $=a$, semi-minor axis $=b$; then area of ellipse is

$$A=\pi ab.$$

For ellipses of small eccentricity, the perimeter is nearly

$$P=2\pi\sqrt{ab}.$$

LESSON LXVI.

If two boys hold a string longer than the distance between them, and a third boy keeping the string stretched and slipping through his fingers, walks around them, what is his path?

How much space will he walk around if the string is 10 feet long, and the boys stand 8 feet apart?

The distance from the end of the minor axis to the foci must be what?

There are two other curves related to the circle and ellipse, called **parabola** and **hyperbola.**

The four are called **conic sections.** Because they can all be made by cutting a right cone properly.

If the cone be cut by a plane parallel to the base, the section is a **circle.**

Cut through the cone obliquely, the section is an **ellipse.**

A section parallel to a side is a **parabola.**

A section parallel to the vertical axis is an **hyperbola.**

A section inclined to axis, not an ellipse or parabola, is an **hyperbola.**

It will not be time lost if the pupil cuts a cone from a potato or turnip, and makes the conic sections.

A **spiral** is a curve that starts from a point and goes round and round, constantly getting farther and farther from the starting-point, never returning into itself.

Of all these curves only the circle is treated of in common geometry, though there is no good reason why the rest should not be studied by pupils of average ability.

LESSON LXVII.

If the sides of a rectangle are 6 and 12, then 2, 3, 4, 5, etc., times 6 and 12, or $\frac{1}{2}$, $\frac{1}{3}$, etc., of 6 and 12 will be the sides of similar rectangles. Why?

Equal multiples of quantities have the same ratio as the quantities themselves.

Let a and b represent the sides of any rectangle that can be imagined.

The area of any rectangle is

$$A = ab.$$

Let m be any number, whole or mixed, or a fraction; then, $m \cdot a$ and $m \cdot b$ will represent the sides of all rectangles similar to the rectangle ab.

The area will be

$$A = m \cdot a \times m \cdot b,$$

or

$$A = m^2 \cdot ab.$$

The ratio of these areas is

$$\frac{m^2 \cdot ab}{ab} = m^2.$$

The ratio of the homologous sides is

$$\frac{m \cdot a}{a} = m;$$

then, the ratio of the areas of similar rectangles is the square of the ratio of their homologous sides.

Or, square the homologous sides a and $m \cdot a$; we have a^2 and $m^2 \cdot a^2$; comparing them, we have

$$\frac{m^2 \cdot a^2}{a^2} = m^2;$$

that is, the areas of similar rectangles and the squares of their homologous sides have a common ratio.

In other words, the areas of similar rectangles are as the squares of their homologous sides.

Substitute in the above statements, surfaces for rectangles, and lines for sides, and they will be true for all similar plane figures.

If a, b, c are the sides of a triangle, what kind of a triangle will it be if $a^2 = b^2 + c^2$? If a, b, c are all equal? If they are unequal? If only two of them are equal?

a, b, c are the sides of any triangle whatever; then $m \cdot a$, $m \cdot b$, $m \cdot c$ are what?

Can either side of a triangle be greater than the sum of the other two?

PART II.

LESSON LXVIII.

How many lines can you draw to pass through one point in a plane? To pass through two points?

We can imagine a line drawn through the center of a sphere.

Lines could be drawn from every point of the surface through the center; that is, lines can be drawn in every possible direction through a point in space. The lines are not limited to a single plane.

Take a call-card; place a point in the center. The point is in the plane; or, we may say, the plane passes through the point.

The plane may be revolved about the point in any direction, being a different plane really every time it moves; and so any number of planes may be made to pass through a point.

Divide an orange or apple into halves. Imagine a line from the stem through the center; put the halves together, pass another plane through the line at right angles to the first plane.

As both planes pass through the center and stem, their intersection must be the line through the center, called an **axis of the sphere.**

Can other planes be passed through the axis?

Then, neither one point, nor two points, nor a line can fix a plane in one position, or, as we say, can determine a plane.

Take a third point on the surface not in the axis. Can you pass more than one plane through it and the center and stem? Then three points not in the same line determine a plane; or, one plane can always be made to pass through three points taken anywhere in space.

For we can imagine any two of them connected by a line; through it pass a plane, and revolve it until it includes the third point.

LESSON LXIX.

Name some planes in the room that intersect in a straight line.

What kind of an angle do the sides of a cube or the room make?

An angle made by two planes is a **dihedral angle.**

As with plane angles, dihedral angles may be acute, right, or obtuse, complementary or supplementary.

For, planes may be parallel, perpendicular, or oblique to each other.

A line also may be parallel, perpendicular, or oblique to a plane.

Draw a diameter across each of two call-cards, each diameter perpendicular to the edges of the card.

Make a dihedral angle, placing the cards with diameters meeting at the edge or intersection of the planes; the angle made by the perpendiculars is the dihedral angle made by the planes.

When two planes come together or intersect, the dihedral angle is the angle made by two perpendiculars to their intersection, one in each plane drawn to the same point.

When a plane intersects two parallel planes, the lines of intersection are parallel. Point out cases in the room, in the cube, etc.

Open the book to different dihedral angles.

When three angles, or three planes, make a solid angle or corner, it is called a **trihedral angle.** Are there any in the room?

In the cube? Mention many.

What kind of angles make the solid angle or corner of the room?

Can you make a solid angle with four right angles?

A solid angle made by three or more planes is a **polyhedral angle.**

Cut four equal equilateral triangles. Can you make a solid of them? How many faces has it? Edges? Polyhedral angles? Are they equal? Are the faces equal? It is a **tetrahedron.**

It is one of five solids called **regular polyhedrons.**

LESSON LXX.

A regular polyhedron has all its solid angles formed by the same number of equal regular polygons. They can each be made with a single piece of paper, properly cut and folded.

Draw the four equal equilateral triangles, and fold them into the tetrahedron. Or,

Draw on thick cardboard an equilateral triangle, side 2 inches.

Join middle points of sides. Cut out the triangle and cut half through the median lines and fold into tetrahedron.

Drawing the faces of a polyhedron in one figure, so as to fold it into the shape of the solid, is developing the surface of the solid.

The next is the **hexahedron,** a regular solid with six equal faces, each a square. Have you ever seen one? How many corners has it? Edges? What is its common name? Develop its surface; cut and fold it.

Draw a rectangle 4 inches by 1 inch. Divide into four squares. On each side of the second square draw another. Cut and fold.

Each corner of the cube is a **tri-rectangular trihedral angle.** For three right angles make the solid angle.

Make a solid angle with four equal equilateral triangles. Make another. Put them together, making a model of a solid.

How many faces has it? Edges? Vertices? Is it a regular polyhedron? It is an **octahedron.**

Draw an oblique parallelogram, sides 3 inches and 1 inch; acute angle 60°. Divide into six equal equilateral triangles, side 1 inch. At each acute angle on the long side, draw an equilateral triangle on base of end triangle. Cut and fold the octahedron.

Make a solid angle with five equilateral triangles, side 1 inch. Make another. Make an oblique parallelogram, sides 5 inches and 1 inch, acute angle 60°. Divide into ten equal equilateral triangles, and cut half through the sides. Make a solid of the three.

How many faces has it? Edges? Solid angles? Is it a regular polyhedron? It is an **icosahedron.**

Draw the parallelogram of ten triangles as above, and on each side five similar triangles. Cut and fold the icosahedron.

LESSON LXXI.

Can you make a solid angle with six equilateral triangles? Why not?

Describe a circle, radius 1 inch. Draw the chords of 72°. What is the figure? Cut it out. With it as a pattern, cut five others.

Also mark out an irregular belt of six pentagons joined by a side.

Make a solid angle with three of the pentagons. Make another with the other three.

Make a polyhedron of the belt and the two solid angles. How many faces has it? Edges? Solid angles?

It is a **dodecahedron.**

Another way to make it: draw a regular pentagon; draw a similar one on each side of the first by producing the sides of the first each way with dotted lines until they intersect; the dotted lines are the bases of isosceles triangles: draw the equal sides equal to side of polygon.

These six polygons are half the surface of the dodecahedron. Cut it out. Cut another and make the solid.

To develop its surface, draw six pentagons as above; take the side of one; produce its adjacent sides with dotted lines to their intersection: the dotted lines are bases of isosceles triangles, whose sides are the sides of a pentagon. Draw them.

On side of the last opposite side first taken, draw another pentagon as before. On the last draw a pentagon on each side. Cut and fold.

Can you make a solid angle using five pentagons? Four? Why not?

Can you make a solid angle with three hexagons?

Then there can be only five regular polyhedrons. Name them. What is a polyhedron?

LESSON LXXII.

A solid bounded by six equal square faces is called what?

What is a cubic inch? Cubic foot? Cubic yard? Cubic meter? Cubic centimeter?

They are units of measurement of solids or space.

The inside dimensions of a box are 10 inches × 8 inches × 6 inches.

How many cubic inches can be placed on the bottom?

How many layers of 80 will fill the box? How many cubic inches, then, will it hold? $10 \times 8 \times 6$.

To find the cubic contents of a box, what must we do?

The volume of a solid is the space it occupies.

The volume of a room is the space it encloses.

Find volume of room, $32 \times 30 \times 14$ feet.

How many cubic inches will it take to make a cube, side 2 inches? Side 3 inches?

Solids bounded by rectangles are called **rectangular parallelopipeds.**

Bricks, blocks of stone, boxes, rooms, etc., are examples.

To find their volume, we multiply the product of length and width by the height or thickness. Thus, if the three dimensions are a, b, c, the volume is

$$V = abc;$$

$$a = \frac{V}{bc}, \quad b = \frac{V}{ac}, \quad c = \frac{V}{ab}.$$

Read the formulas in words.

$m \cdot a$, $m \cdot b$, $m \cdot c$, will be the dimensions of any similar solid, m being the common ratio of the homologous sides of the solids.

The volume is $V = m \cdot a \times m \cdot b \times m \cdot c$,

or $V = m^3 \cdot abc$.

Comparing volumes,

$$\frac{m^3 \cdot abc}{abc} = m^3;$$

that is,

The ratio of the volumes of similar parallelopipeds is the cube of the common ratio of two homologous lines of the solids.

The same is true of all similar solids.

LESSON LXXIII.

Compare the surfaces of three cubes, sides of which are 1 inch, 2 inches, and 3 inches. Volumes?

Which has the greatest surface in proportion to its volume?

Compare two rectangular parallelopipeds, similar, and ratio of homologous sides, 2.

Cut an orange in halves.

Each half is a hemisphere and is bounded by two surfaces; a plane one which is a great circle of the whole sphere or orange, and a spherical surface that is just twice the area of the plane surface.

Cut a half into quarters.

How many surfaces has the quarter?

The area of the spherical surface of the quarter is the area of a great circle.

Area of each plane surface is .....?

Cut a quarter into eighths by a plane through the axis as before.

The three surfaces of the eighth are of equal area.

To what is each equal?

As the sphere is divided into smaller parts, we see the whole surface increases; and we must conclude that

The smaller the body, or solid, the greater its surface in proportion to its volume.

Take two spheres, radii 1 inch and 2 inches.

Their surfaces are as $1^2 : 2^2$, or $1 : 4$.

Their volumes are as $1^3 : 2^3$, or as $1 : 8$.

LESSON LXXIV.

Cut a sector from a circle. What is its area?

Bring the radii and ends of the arc together. It makes a **cone.**

How would you find the convex surface of a cone?

The radius of the sector is the slant height of the cone.

The base is a circle, radius r;

perimeter of base $=2\pi r$,
area of base $=\pi r^2$,
convex surface of cone $=2\pi r\times$ slant height,
entire surface $=\pi r^2+2\pi r\times h$,
or $\pi r(r+2h)$,
$h=$ slant height.

Describe a semicircle with radius $\frac{1}{8}$ foot.

Cut from it a sector of 170°.

Form a cone with it by lapping the circumference at the base 10° or about $\frac{3}{16}$ of an inch.

(Have the paper firm and glue the edges strong.)

Make a cylinder open at both ends, 1 inch in diameter and 1 inch high. Have a card for loose top and bottom to the cylinder.

Find how many times the cylinder will hold the contents of the cone. Use fine sand.

Put an inch sphere into the cylinder.

The cylinder will require one volume of the cone to fill both ends. Compare the three.

The volumes of a cone, sphere, and cylinder, of the same height and diameter, are to each other as 1 : 2 : 3.

The volume of a cone is $\frac{1}{3}$ of the volume of a cylinder of the same base and altitude.

LESSON LXXV.

A point may trace a line. A line may move over a surface.

So we can imagine a surface to move through the space occupied by a solid; the side of a rectangle, for example.

Some familiar solids are **solids of revolution.**

If a right triangle revolves about a leg, what solid is generated?

It is a right cone, because its axis is perpendicular to the base.

If a rectangle revolves about a side, it generates a **cylinder.**

A semicircle revolved on a diameter generates a **sphere.**

A semi-ellipse revolved on its major axis generates an **ellipsoid** or **prolate spheroid.**

Revolved on its minor axis, it generates an **oblate spheroid.**

Cut a paper rectangle 2 inches by 3 inches. What is its area? Make a cylinder of it. What is the diameter of the base? Length or height? Area of both bases? Convex surface?

If a circle moves on a line perpendicular to its plane, it traces a **right cylinder.**

The volume of the cylinder is evidently the area of the base by height of cylinder.

$$V = \pi r^2 \times h.$$

When diameter and height are equal, we have for

volume of cylinder, $V=2\pi r^3$;
inscribed sphere, $V=\frac{4}{3}\pi r^3$;
inscribed cone, $V=\frac{2}{3}\pi r^3$.

All solids similar to the above must have $m\cdot r$ for radius of circle. And their volumes would be

$$V=2\pi m^3\cdot r^3, \text{ etc.}$$

Compared with the first,

$$\frac{2\pi m^3\cdot r^3}{2\pi r^3}=m^3;$$

that is,

The ratio of the volumes of similar solids is the *cube* of the common ratio of their homologous lines. Or,

Similar solids are as the cubes of their homologous lines.

1. A 3-inch cannon-ball weighs 10 pounds; a 6-inch weighs how much?

2. Compare the volumes of two oranges, 2 inches and $2\frac{1}{2}$ inches diameter.

◆

LESSON LXXVI.

If a surface moves at right angles to its plane, it traces a **right prism.**

The volume of the prism is

Area of base × height of prism.

$$V=A\times h.$$

The prism is called triangular, square, rectangular, or polygonal, according as the base is a triangle, square, rectangle, or polygon.

If the base is a circle, what is the prism called?

A pyramid is cone-shaped, and is called a **triangular pyramid,** a **square pyramid,** etc., according as its base is a triangle, square, etc., as with the prism.

What must be the shape of one of its sides?

How would you find its surface, including the base?

$$S = \text{base} + \text{perimeter of base} \times \tfrac{1}{2} \text{ slant height.}$$

A right cone, pyramid, or cylinder is one whose axis is perpendicular to the base. The axis is a line from the vertex to middle point of the base.

Volume of cone or pyramid is

$$V = B \times \tfrac{1}{3} \text{ altitude,}$$

or

$$V = B \times \tfrac{1}{3} h.$$

The altitude of an oblique cone or pyramid is the perpendicular distance of the vertex from the base.

The altitude of an oblique cylinder is the perpendicular distance between its parallel bases.

Its volume, $V = b \times h.$

Ex. 1. Find volume of right cone; radius of base 2 feet; height 12 feet. Slant height. Convex surface. Entire surface.

Ex. 2. Find volume of a triangular prism; height 10 feet; base an equilateral triangle; side 4 feet. Area of lateral faces. Area of bases. Entire surface.

LESSON LXXVII.

We found the surface of the sphere equal to $4\pi r^2$; or, four times the area of the generating circle,

$$S = 4\pi r^2.$$

The volume of the sphere is $\frac{4}{3}\pi r^3$.

Suppose the sphere to be made up of very small pyramids; that is, their bases making the surface of the sphere exceedingly small, with all their apexes at the center.

Then, the sum of the bases of the pyramids is the surface of the sphere, $4\pi r^2$; and the height of each pyramid is r.

Therefore the volume of the sphere is

$$4\pi r^2 \times \tfrac{1}{3} r,$$

or

$$V = \tfrac{4}{3}\pi r^3.$$

Ex. 1. Suppose the earth a sphere with a radius of 4000 miles. Find its surface and its volume.

Ex. 2. Two globes have diameters, 4 feet and 6 feet. Compare surfaces and volumes.

If a semi-ellipse revolves around its minor axis $(2b)$, the volume of the oblate spheroid is $V = \frac{4}{3}\pi a^2 b$.

Revolved around its major axis $(2a)$, the volume of the prolate spheroid is $V = \frac{4\pi}{3} ab^2$.

Ex. 1. Axes of ellipse are 4 and 6. Volume of prolate spheroid is .....? Volume of oblate spheroid is .....?

LESSON LXXVIII.

If a circle revolves about its diameter (*Pp*) and points *A*, *B*, *C*, *D*,. . ., *P*, be taken in one quadrant, and points *b*, *c*, *d*, . . ., *p*, be marked in the other quadrant, what will be the paths described by these points?

Which point describes the largest circle? *A* is equally distant from the poles (*Pp*) of the axis, and the circle it describes is called a **great circle** of the sphere or equator, with reference to the poles *P* and *p*.

All other points from *A* to *P* and from *A* to *p*, describe smaller and smaller circles on the sphere, until we get to *P* and *p*, which are simply points or the ends of the axis of revolution.

In speaking of the earth, we call them (*p*, *P*) the north pole and the south pole.

Circles made by points between *A* and *P*, or *A* and *p*, are called parallel circles of latitude, each divided into 360° of longitude.

Great circles through the poles are meridians, and each divides the globe into hemispheres.

Latitude is measured on a meridian from the equator to the pole, so there can be only 90° of latitude north, and 90° of south latitude.

Longitude is measured on the equator or a parallel, from a meridian 180° each way, or half round the sphere. And is called east or west longitude according to its direction.

The shortest distance between two points on a sphere is measured on a great circle through them.

If we call radius of the earth 4000 miles, how long is a degree of latitude? 1 minute? 1 second?

How long is a degree of longitude on the equator?

◆

LESSON LXXIX.

If a and b represent two numbers, the square root of their product $\sqrt{ab}$ is called the mean proportional between a and b.

Thus, $4 \times 9 = 36$, $\sqrt{36} = 6$, and 6 is the mean proportional between 4 and 9; that is, $4 : 6 :: 6 : 9$.

If the upper part of a cone or a pyramid is cut off by a plane parallel to the base, the lower part is called the **frustum** of a cone or pyramid.

The section parallel to the base is a circle or polygon similar to the base.

Surface of the frustum is

$$S = \text{bases} + \text{lateral faces}.$$

Volume is $$V = (B + b + \sqrt{Bb}) \times \frac{h}{3};$$

that is, the volume of the frustum is equal to the sum of the areas of the bases and their mean proportional, multiplied by $\frac{1}{3}$ of its height.

Thus height of frustum of a cone is 20 feet, radii of bases 4 and 9 feet. Volume?

Area of lower base $= 9^2 \times \pi$

Area of upper base $= 4^2 \times \pi$

Mean proportional $\sqrt{(4 \times 9)} = 6 = 6^2 \times \pi$

Adding areas $= 133 \times \pi$

Therefore, volume is

$$133 \times \pi \times \tfrac{20}{3} = \quad ?$$

Ex. 1. Find entire surface of frustum of a pyramid; lower base 9 feet square, upper 4 feet, height 10 feet. Find volume.

* * *

LESSON LXXX.

Ex. 1. Height of right cylinder is 8 feet, radius of bases 2 feet.

a. Find area of bases.
b. Convex surface.
c. Entire surface.
d. Volume.

Ex. 2. Find volume of the hexagonal prism that will just go into the above cylinder.

a. Its lateral surface.
b. Area of bases.
E. Entire surface.

Ex. 3. Find volume of cone to fit above cylinder.

a. Its convex surface.
b. Its entire surface.

Ex. 4. Find volume of square pyramid that will go into the cylinder.

a. Area of base.
b. Convex surface.

LESSON LXXXI.

THEOREMS AND PROBLEMS.

A **theorem** is a proposition to be demonstrated.

A **problem** is something to be done with the ruler and compasses.

1. When two straight lines intersect, if one angle is a right angle, the other three are right angles.

2. In a triangle, the greater side is opposite the greater angle.

3. The perpendicular is the shortest distance to a line from a given point.

4. The lines bisecting two supplementary adjacent angles are perpendicular to each other.

5. What is the locus of a point always equidistant from the extremities of a given line?

6. If two right triangles have an acute angle and one leg in each equal, the triangles will be equal.

7. What is the length of a line from the right angle to middle of hypotenuse?

8. Bisect a line; an angle.

9. Trisect a right angle, opening the compasses only once. Radius is chord of 60°.

10. Make a square double the area of a given square.

11. The locus of a point, the square of the distances from which to two fixed points is constant, is what?

12. The sum of all the angles of a quadrilateral equals what?

13. Sum of the angles of a pentagon? Hexagon? Any regular polygon?

14. Sum of the exterior angles, formed by producing sides of a polygon, is what?

15. Make a rhombus having both diagonals given.

16. Construct a rectangle, the perimeter and one side being given.

17. How would you go from the middle of one end of a room to the opposite point, touching a side, and make the shortest journey?

18. The area of a triangle is equal to the square root of the product of half the sum of its sides, and the difference between the half sum and each side.

Sides are 9, 12, and 15; required the area.

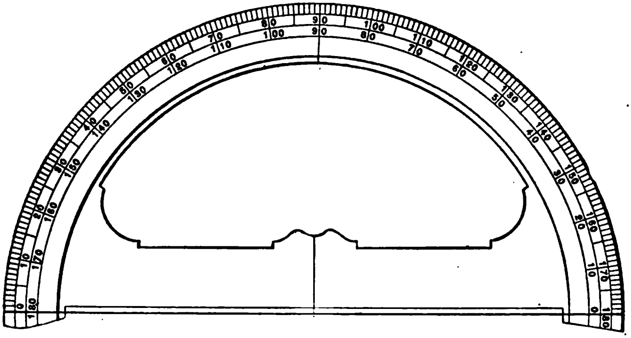
19. Diameter of circle is 100. Find area of inscribed square; of inscribed hexagon; of inscribed equilateral triangle; area of similar circumscribing figures.

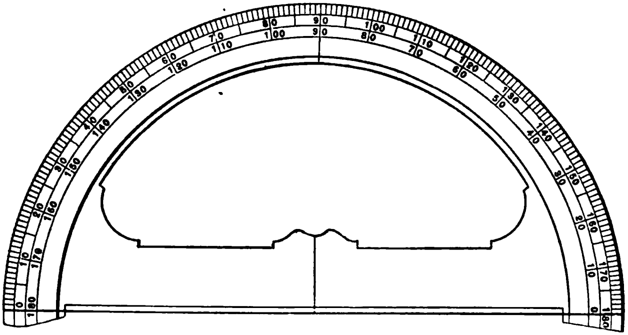
20. Draw, freehand, triangle with sides a, b, and c:
(1) When they are unequal. (3) When $a=b$.
(2) When they are equal. (4) When $a^2+b^2=c^2$.

Give a, b, and c any values.

21. Construct:

a, $a-b$, ab, $(a+b)(a-b)$,
$a+b$, a^2, $(a+b)(a+b)$, $(a-b)(a-b)$.





The pupil can make a protractor by pasting one of the above on a stiff piece of paper and carefully cutting it out.

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